AVERAGE MARKS SCALING

§1. Introduction

In Western Australia a wide range of courses is available for senior secondary students. For each course there is an external examination taken at the end of Year 12. The examination results are then combined with school assessments. This process involves standardisation and moderation. The way in which this is done is described on the Curriculum Council website at www.curriculum.wa.edu.au.

Thus, for each student a composite unscaled mark is generated for each course taken. However, the cohort of students taking a particular course $P$ may be academically more able than the cohort of students taking course $Q$. This means that a composite unscaled mark of, say, 60 in course $P$ represents a higher level of attainment than the same mark in course $Q$. Since marks in different courses are added to form a Tertiary Entrance Aggregate (TEA), equity considerations require that the marks in courses $P$ and $Q$ should be scaled. This means that the marks in course $P$ should be scaled up relative to the marks in course $Q$ (or the marks in course $Q$ scaled down.) Of course, the situation is more complex than just establishing the appropriate relativity of the marks in courses $P$ and $Q$. In fact there are many courses that students can take and so it is the relativities of all courses which scaling needs to address.

The key concept is to generate a measure of academic ability, based on students’ achievements, for the cohort of students taking a particular course. The scaling method used in Western Australia is based on the premise that the best measure of an individual student’s academic performance is that student’s average scaled mark across all courses taken. Suppose that average is denoted by $t_i$. That is to say, for a particular student $i$ define $t_i$ to be the average scaled mark obtained by student $i$ across all courses taken by student $i$.

Now consider the cohort of students taking a particular course $j$. For each of these students there will be an average scaled mark $t_j$. The average of the $t_j$ across all the students taking course $j$ will be denoted by

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1 The terminology ‘course’ is used in the new structure introduced by the Curriculum Council. Previously the term ‘subjects’ was used to describe what have now become courses.

2 This is often referred to as the final combined mark (for WACE courses, this is after adjustment between stages and the stage 3 increment has been applied as appropriate).

3 Average Marks Scaling has been used in New South Wales for many years. The main architect of that system was Professor E. Senata, Department of Mathematical Statistics, University of Sydney. The New South Wales system was adapted for use in Western Australia by Dr M.T. Partis, Director, Secondary Education Authority, in 1997 and introduced in 1998. Prior to that the Australian Scaling Test was used as the anchor variable in the scaling process.
That is to say, for a particular course \( j \) define \( T_j \) to be the average of the average scaled marks \( t_i \) obtained by all students taking course \( j \). The Average Marks Scaling (AMS) process uses \( T_j \) as a proxy measure\(^4\) of the academic ability of the cohort of students taking course \( j \).

At the end of the process the value of \( T_P \) for course \( P \) and the value of \( T_Q \) for course \( Q \) can be calculated. The mean\(^5\) mark for the cohort of students taking course \( P \) is then scaled to \( T_P \), whilst the mean mark for the cohort of students taking course \( Q \) is scaled to \( T_Q \). This gives the appropriate relativity between the two courses. Adjustments are also made to the standard deviations of the course distributions, but these turn out to be minor compared to the adjustments to the means.

It is worth stressing the main feature of Average Marks Scaling. The description above seems to suggest that the scaled marks need to be known in order for scaling to be carried out. However, at the heart of the process is the equating of the average scaled mark in course \( j \) and the anchor variable \( T_j \). The mathematics involved is set out below, but the key equation

\[
\text{Average scaled mark in course } j \text{ of the cohort of students taking course } j = \text{The anchor variable } T_j \text{ for the course } j \quad \ldots \quad (1)
\]

provides the central focus of the analysis.

To clarify the concepts involved it is worth considering a numerical example. Suppose that student \( i \) has a scaled mark of 63 in course \( j \). In what follows this scaled mark will be denoted by \( y_{ij} \).

That is to say,

\[ y_{ij} = 63 \]

Suppose that student \( i \) takes four other courses obtaining scaled marks in those of 47, 71, 58 and 66. Then the average of this student's five scaled marks will be 61.

That is to say,

\[ t_i = 61 \]

It is important to note that, in this instance, the values of \( y_{ij} \) and \( t_i \) are not the same. This is because they are measuring different things. \( y_{ij} \) is measuring the student's performance in course \( j \), whilst \( t_i \) is measuring the student's performance across five courses.

Now consider the cohort of all students taking course \( j \). For each of these students there will be corresponding values of \( y_{ij} \) and \( t_i \). Hence, the average values of \( y_{ij} \) and \( t_i \) for the whole cohort can be

\(^4\) \( T_j \) is sometimes described as the anchor variable for the scaling process.

\(^5\) Throughout this document the terms ‘mean’ and ‘average’ will be regarded as synonymous.
determined. The scaling process is designed to make these averages identical. In simple terms the unscaled marks in course $j$ are moved up or down to achieve this outcome.

The adjustment of the composite unscaled marks to scaled marks uses a linear conversion, the details of which are explained below. It is important to note the following points:

- the AMS scaling process preserves the ranking of students and the shape of the distribution in each course;
- the mean scaled score across all courses and all students (the global mean) is predetermined, and currently set at 60$^6$.
- the standard deviation parameter for the AMS process is currently set at 13$^7$.

§2. Re-standardisation using z-scores

Although examination marks and school assessments are standardised at an earlier stage of the process, it is mathematically convenient to standardise again in such a way that the composite unscaled mark distribution for each course has a mean of 0 and a standard deviation of 1.

Let $w_{ij}$ be the composite unscaled mark for student $i$ in course $j$. For each course $j$ let $\mu_j$ and $\sigma_j$ be the mean and standard deviation of the $w_{ij}$.

Now put

$$z_{ij} = \frac{w_{ij} - \mu_j}{\sigma_j}.$$ 

The $z_{ij}$ values generated in this way are often referred to as $z$-scores. By definition it follows that the $z$-scores for each course will have a mean of 0 and a standard deviation of 1.

The AMS process uses the $z_{ij}$ values defined above to generate scaled marks $y_{ij}$. The conversion to scaled marks can be regarded as a three-part process.

First, add 60 to restore the overall mark distribution to the predetermined global mean.

Second, add a term $d_j$ for each course $j$ which determines whether the marks in that particular course are scaled up or down relative to the global mean. This means that some $d_j$ will be positive and some negative.

Third, add a term $13c_jz_{ij}$, which alters the standard deviation for each course from 1 to $13c_j$. The value of 13 is predetermined to produce an appropriate global standard deviation. The $c_j$ will be calculated for each course $j$, but in practice the values turn out to be always close to 1.

$^6$ The scaling mean was 58 prior to 2009
$^7$ The scaling standard deviation was 14 prior to 2010

The combination of the three steps outlined above gives rise to the following equation:

The scaled mark $y_{ij}$ for student $i$ in course $j$ is given by

$$y_{ij} = 60 + d_j + 13c_jz_{ij}.$$
The parameters \( d_j \) and \( c_j \) need to be evaluated for each course \( j \). It should be emphasised that whilst equation (2) gives an \textit{algebraic} definition of the scaled marks \( y_{ij} \), the \textit{arithmetical} values of the \( y_{ij} \) can only be calculated after the parameters \( d_j \) and \( c_j \) have been determined. The way in which the parameters \( d_j \) and \( c_j \) are evaluated is set out below.

\[ y_{ij} = 60 + d_j z_{ij} \ldots (2) \]

\[ \text{§3. Calculating averages} \]

In this section the technical details of working out the average of the average scaled marks, that is to say \( T_j \) for each course \( j \), are developed. For this purpose it is useful to introduce a function \( \alpha_{ij} \) which depends on whether a particular student is taking a course or not. Define

\[
\alpha_{ij} = \begin{cases} 
1 & \text{if student } i \text{ takes course } j; \\
0 & \text{if student } i \text{ does not take course } j. 
\end{cases}
\]

Let \( n \) be the total number of students taking the examinations\(^8\).

Let \( m \) be the total number of courses available in the examinations.

The number of students \( n_j \) taking course \( j \) is then given by

\[ n_j = \sum_{i=1}^{n} \alpha_{ij} \]

The number of courses \( m_j \) taken by student \( i \) is given by

\[ m_i = \sum_{j=1}^{m} \alpha_{ij} \]

For a particular student \( i \) the \textit{average scaled mark}, denoted by \( t_i \), over all courses taken by that student is given by

\[ t_i = \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} y_{ik} \]

\(^8\) In practice a subset of the total number of students, known as the \textit{scaling population}, is used. The intention is to exclude, for example, students taking only one course.

For all the students taking a particular course \( j \) the \textit{average of the average scaled marks}, denoted by \( T_j \), is given by

\[ T_j = \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} t_i \]
\[
\sum_{i=1}^{n} \alpha_{ij} \left( \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} y_{ik} \right) = \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} \left( \frac{1}{m_i} \sum_{k=1}^{m} \alpha_{ik} (60 + d_k a_{jk} + 13 c_k z_{ik}) \right)
\]

The significance of \( t_i \) and \( T_j \) was explained in §1. \( T_j \) is the anchor variable for the scaling process.

**§4. Matrix representation**

In §3 the formula for \( T_j \) was derived. This generates \( m \) equations, corresponding to the values of \( j \) running from 1 through to \( m \). The next stage in the process is to recast these \( m \) equations in a matrix format.

From the previous section it follows that

\[
T_j = \sum_{k=1}^{m} \left( 60 b_{jk} + d_k b_{jk} + 13 c_k a_{jk} \right)
\]

where

\[
b_{jk} = \frac{1}{n_j} \sum_{i=1}^{n} \frac{\alpha_{ij} \alpha_{ik}}{m_i} \quad \text{and} \quad a_{jk} = \frac{1}{n_j} \sum_{i=1}^{n} \frac{\alpha_{ij} \alpha_{ik} z_{ik}}{m_i}
\]

This leads to the matrix equation

\[
\mathbf{T} = 60 \mathbf{B} \mathbf{1} + \mathbf{B} \mathbf{d} + 13 \mathbf{A} \mathbf{c} \quad \ldots \ldots \quad (3)
\]

where

\( \mathbf{T} \) is the \( m \times 1 \) column vector \( [T_j] \);

\( \mathbf{A} \) and \( \mathbf{B} \) are the \( m \times m \) matrices \( [a_{jk}] \) and \( [b_{jk}] \), respectively;

\( \mathbf{1} \) is the \( m \times 1 \) column vector with each entry equal to 1;

and \( \mathbf{c} \) and \( \mathbf{d} \) are the \( m \times 1 \) column vectors \( [c_{jk}] \) and \( [d_{jk}] \), respectively.

Analysis of the \( b_{jk} \) which are the elements of matrix \( B \) then gives

\[
B \mathbf{1} = \mathbf{1}
\]

Hence, equation (3) simplifies to

\[
\mathbf{T} = 60 \mathbf{1} + \mathbf{B} \mathbf{d} + 13 \mathbf{A} \mathbf{c} \quad \ldots \ldots \quad (4)
\]

**§5. Average scaled mark in course \( j \)**
From equation (2) the scaled mark $y_{ij}$ for student $i$ in course $j$ is given by

$$y_{ij} = 60 + d_j + 13c_jz_{ij}.$$ 

Hence, for a particular course $j$, the average of the $y_{ij}$ is given by

$$\frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij}y_{ij} = \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij}(60 + d_j + 13c_zz_{ij})$$

$$= 60 + d_j + 13c_j \left( \frac{1}{n_j} \sum_{i=1}^{n} \alpha_{ij} z_{ij} \right)$$

$$= 60 + d_j$$

since the $z$-scores for course $j$ have a mean of 0.

From equation (1) this gives

$$T_j = 60 + d_j$$

Hence,

$$T_j = 60 \uparrow + d_j$$ $\ldots \ldots \ (5)$$

From equations (4) and (5) it follows that

$$60 \uparrow + d_j = 60 \uparrow + B \downarrow d + 13A \downarrow c$$

Simplifying this matrix equation gives

$$\begin{array}{c}
(I - B) d = 13A c \\
\end{array} \ldots \ldots \ (6)$$

where $I$ is the $m \times m$ identity matrix.

### §6. Calculation of the key parameters

In matrix equation (6) the only unknowns are the column vectors $c$ and $d$. The $c_j$ entries which make up the column vector $c_j$ can be thought of as the standard deviations for each course $j$. This can be evaluated by considering the $z$-scores for all students taking course $j$.

The standard deviation $c_j$ for the $z$-scores in course $j$ is given by a complicated (but standard) formula, namely

$$c_j^2 = \frac{1}{n_j} \left[ \sum_{i=1}^{n} \alpha_{ij} \left( \frac{\sum_{k=1}^{m} \alpha_{ik} z_{ik}^2}{\sum_{k=1}^{m} \alpha_{ik}} \right) \right] - \frac{1}{n_j} \left[ \sum_{i=1}^{n} \alpha_{ij} \left( \frac{\sum_{k=1}^{m} \alpha_{ik} z_{ik}}{\sum_{k=1}^{m} \alpha_{ik}} \right) \right]^2$$
The arithmetical values for the $c_j$ can now be substituted into equation (6). Solving equation (6) gives the values of $d_j$.

Equation (2) now allows the scaled marks $y_{ij}$ to be calculated for all students in all courses.

§7. Example showing how a student's scaled mark is calculated

Consider a student whose unscaled combined mark in course $j$ is 65.73. For the cohort of students taking course $j$ suppose that the mean and standard deviation of the unscaled combined marks are 59.51 and 12.20, respectively.

Now derive the $z$-score corresponding to the student’s unscaled mark of 65.73. This is

$$z_{ij} = \frac{w_{ij} - \mu_j}{\sigma_j} = \frac{65.73 - 59.51}{12.20} = 0.51$$

For course $j$ suppose that the scaling parameters turn out to be $c_j = 1.02$ and $d_j = 5.22$. Then equation (2) gives the scaled mark $y_{ij}$ as

$$y_{ij} = 60 + d_j + 13c_j z_{ij} = 60 + 5.22 + 13(1.02)(0.51) = 71.98$$

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